GROUP ANALYSIS OF THE HEAT-CONDUCTION EQUATION.

1. INVARIANT SOLUTIONS

N. M. Tsirel'man

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Invariant solutions of the heat-conduction equation are constructed in terms of displacements of the isotherms.

It is generally known that the application of the algorithms of group analysis to the case when the mathematical model leads to a partial differential equation (or a system of such equations) gives nontrivial results [1-4]. The problem of interest here is the nonlinear heat-conduction equation

$$\frac{\partial T}{\partial \tau}=\frac{\partial}{\partial x}\left[f(T)\frac{\partial T}{\partial x}\right], \tag{1}$$

for which Ovsyannikov [1] obtained invariant solutions. The substitution $T = \int c\rho(\tilde{T})d\tilde{T}$ converts (1) to the equation

$$c\rho(\tilde{T})\frac{\partial \tilde{T}}{\partial \tau}=\frac{\partial}{\partial x}\left[\lambda(\tilde{T})\frac{\partial \tilde{T}}{\partial x}\right], \tag{2}$$

where $c\rho(\tilde{T})$ and $\lambda(\tilde{T})$ are the heat capacity and thermal conductivity, respectively. When we make a change of variables [5] in Eq. (1) to the position x of the isotherms T = const (as a function of the time τ), Eq. (1) takes the form

$$x'_\tau = f(T)x''_{TT}(x'_T)^{-2} - f'(T)(x'_T)^{-1}. \tag{3}$$

Equation (3) then describes the unsteady heat conduction in bodies both with and without a phase transition in terms of the displacements of the isotherms.

The function f(T) can be arbitrary, and therefore (1) is a general nonlinear equation. However, in (3) the nonlinearity is due to the presence of the factors $(x_T')^{-2}$ and $(x_T')^{-1}$ and is therefore regular. This facilitates the obtaining of invariant solutions in a series of cases.

Note that for (3) the system of one-parameter subgroups

$$X_1=\frac{\partial}{\partial \tau},\ X_2=\frac{\partial}{\partial x},\ X_3=2\tau\frac{\partial}{\partial \tau}+x\frac{\partial}{\partial x} \tag{4}$$

established in [1] for arbitrary f(T) continues to hold, and there is the additional operator

$$X_4=-\tau\frac{\partial}{\partial \tau}+\frac{\partial}{\partial T} \tag{5}$$

for the special case f(T) = exp T.

The power law $f(T) = T^{2m}$ ($m \neq -2/3$) gives the additional operator

$$X_4=mx\frac{\partial}{\partial x}+T\frac{\partial}{\partial T}, \tag{6}$$

and when m = −2/3 we have the operator

$$X_5=x^2\frac{\partial}{\partial x}-3xT\frac{\partial}{\partial T}. \tag{7}$$

We write out the invariant solutions to Eq. (3).

1. The function f(T) is arbitrary; the subgroups X_1, X_2, X_3, $X_1 + X_2$ give the invariant solutions

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$$x = v(T),\ x = \text{const},\ x = \sqrt{\tau}\, v(T),\ x = \tau + v(T). \tag{8}$$

2. We have f(T) = exp T; use of the subgroups $X_2 + X_4$, $\gamma X_3 + X_4$, $X_1 + X_3 + 2X_4$ (γ is an arbitrary constant) gives the following invariant solutions

$$x = T + v(T + \ln \tau),\ x = v[\exp(T)\,\tau^{1-2\gamma}]\,\tau^{\gamma/(2\gamma-1)}, \quad x = v(T - 2\tau)\exp\tau. \tag{9}$$

in addition to the solutions (8).

3. The function $f(T) = T^{2m}$ ($m \neq -2/3$); the subgroups $X_1 + X_4$, $\gamma X_3 + X_4$, $X_2 - mX_3 + X_4$ lead to the solutions

$$x = \exp(m\tau)\, v[T\exp(-\tau)],\ x = v(T\tau^{-1/2\gamma})\,\tau^{(\gamma+m)/2\gamma}, \quad x = \ln T + v(x + \ln \tau/2m). \tag{10}$$

in addition to (8).

4. The function $f(T) = T^{4/3}$; the subgroups X_5, $X_2 + X_5$, $X_3 + X_5$ give the solutions

$$x = T^{-3}v(\tau),\ x = T^{-3}v(\tau - 1/x),\ x = T^{-3}v[\tau^{1/2}x/(x+1)] \tag{11}$$

in addition to the solutions obtained in (10) (except for the last one).

1. Let f(T) be an arbitrary function. Then the operator X_1 corresponds to $\partial/\partial\tau$ and x are invariants. Substituting a solution of the form x = v(T) in (3), we obtain

$$\frac{f'(T)}{v'(T)} = f(T)\frac{v''(T)}{[v'(T)]^2}, \tag{12}$$

and hence

$$\frac{f'}{f} = \frac{v''}{v'} \text{ and } \int f(T)\,dT = Av(T) + B,$$

or

$$x = v(T) = \left[\int f(T)\,dT - B\right]/A. \tag{13}$$

Hence in this case x = v(T) coincides with the well-known steady solution.

2. Let f(T) be an arbitrary function. Then the operator X_2 corresponds to $\partial/\partial x$ and the invariants are τ and T. In this case there are no invariant solutions to (3) because there are no invariants which are functions of the dependent variable (i.e. x).

3. Let f(T) be an arbitrary function. The operator $X_3 = 2\tau\,\partial/\partial\tau + x\,\partial/\partial x$ generates the invariants T and x^2/τ, and the invariant solution takes the form $x = \sqrt{\tau}\, v(T)$. With the use of (3) we obtain the equation

$$v''(T) - \{f'(T)v'(T) + v(T)[v'(T)]^2/2\}/f(T) = 0, \tag{14}$$

which can be reduced to the form

$$\left(\frac{v'}{f}\right)' = \frac{1}{2}v\left(\frac{v'}{f}\right)^2 \text{ or } \left(\frac{f}{v'}\right)' = -\frac{1}{2}v.$$

The substitution $u = \int v(T)\,dT/\sqrt{2}$ leads to the equation

$$uu'' + f(T) = 0, \tag{15}$$

whose solution can be expanded in a series in T:

$$u = \sum_{k=0}^{\infty} u_k T^k$$

If we represent f(T) in the form $f(T) = \sum_{k=0}^{\infty} f_k T^k$ and take into account that $u'' = \sum_{i=0}^{\infty}(i+1)(i+2)u_{i+2}T^i$, then upon equating the coefficients of powers of T^k to zero we obtain

$$\sum_{i=0}^{k}(i+1)(i+2)u_{i+2}u_{k-i} + f_k = 0. \tag{16}$$

The recursion relation for the u_k has the form

$$(k+1)(k+2)u_0u_{k+2} = f_k - \sum_{i=0}^{k-1}(i+1)(i+2)u_{k-i}u_{i+2}.$$

4. Let f(T) be an arbitrary function; the invariants corresponding to the operator $X_1 + X_2 = \partial/\partial\tau + \partial/\partial x$ are T and $x - \tau$, and the invariant solution is of the form $x - \tau = v(T)$. It is not difficult to show that

$$x-\tau=\int [f(T)\,dT/(A-T)]+B. \tag{17}$$

Equation (3) is invariant to translations in τ, and therefore the constant B can be taken to be zero.

5. Let f(T) be equal to exp T, then the operator $X_2 + X_4$ corresponds to $\partial/\partial x - \tau\,\partial/\partial\tau + \partial/\partial\tau$ and generates the invariants $\tau \exp x$, $\tau \exp T$, $T - x$. The invariant solution is obtained in the form

$$x=T+v(\tau\exp T)=T+v(\mu), \tag{18}$$

and substitution of this relation into (3) gives, upon equating u to v', an Abel equation of the first kind

$$\mu^2u'+(2\mu-1)u-2\mu u^2-\mu^2u^3+1=0 \tag{19}$$

It can be shown that $u = -\mu^{-1}$ is a particular solution of (19). The substitution $u = \mu^{-1} + \mu^2/z$ brings (19) to the form

$$z'+\mu^4/z-\mu=0. \tag{20}$$

This is an Abel equation of the second kind which has a whole set of exact solutions, but we omit the calculations here as they are complicated.

An analysis based on the "traditional" equation (1) does not lead to a determination of $v(\mu)$, since it would be necessary to integrate the equation

$$v'\exp x=\exp(x+v)\,[1+2v'\tau\exp x+(v')^2\tau^2\exp(2x)+\tau v'\exp x+\tau^2v''\exp(2x)],$$

and upon the substitution $\mu = \tau \exp x$ this equation simplifies only slightly:

$$v'=\{1+3\mu v'+\mu^2[(v')^2+v'']\}\exp v.$$

6. Let f(T) be equal to exp T; the operator X_4 corresponds to $-\tau\,\partial/\partial\tau + \partial/\partial T$ ($\gamma = 0$), the invariants are x and $\tau \exp T$. The invariant solution has the form $x = v(\mu) = v(\tau \exp T)$, and from (3) we obtain the equation

$$v''=(v')^3. \tag{21}$$

Letting $v' = w$, we can reduce (21) to the form $w' = w^3$ and its solution is $w = v' = \pm[2(A - \mu)]^{-1/2}$. It then follows that

$$v=\pm(C-2\mu)^{1/2}+B, \tag{22}$$

or

$$x=\pm(C-2\tau\exp T)^{1/2}+B,$$

or

$$T=\ln[C-(x-B)^2/2\tau].$$

Because (8) is invariant to displacements in x, one can put B = 0.

7. Let f(T) be equal to exp T. Then the operator

$$\gamma X_3+X_4=(2\gamma-1)\tau\frac{\partial}{\partial\tau}+\gamma x\frac{\partial}{\partial x}+\frac{\partial}{\partial T}\ (\gamma\neq 0)$$

generates the invariants $x\tau^{-\gamma/(2\gamma-1)}$, $\exp(T)\tau^{-1/(2\gamma-1)}$. The invariant solution is

$$x=v\,[\exp(T)\,\tau^{-1/(2\gamma-1)}]\,\tau^{\gamma/(2\gamma-1)}=v(\mu)\,\tau^{\gamma/(2\gamma-1)}.$$

The function $v(\mu)$ satisfies the equation

$$v''+[(v')^3-\gamma v\,(v')^2/\mu]/(2\gamma-1)=0 \tag{23}$$

and the substitution $\mu_1 = \mu(v)/(2\gamma - 1)$ brings this to the form

$$\mu_1\mu_1''-\mu_1+\gamma v\mu_1'=0, \tag{24}$$

and we can obtain the particular solution

$$\mu_1(v)=(1-2\gamma)\,v^2/2.$$

The general series solution of (24) is complicated and we do not write it out here.

8. Let $f(T) = T^{-4/3}$; then the operator X_5 corresponds to $x^2\,\partial/\partial x - 3xT\,\partial/\partial T$, and the invariants are τ and $xT^{1/3}$. The invariant solution has the form $x = T^{-1/3}v(\tau)$. With the help of (3) we then obtain $v' = 0$, $v = C$, and this means that

$$xT^{1/3}=C. \quad (25)$$

The solution (25) corresponds to the steady state temperature distribution in a rod.

9. Let $f(T) = T^{-4/3}$. Then the operator $X_2 + X_5 = (1 + x^2)\,\partial/\partial x - 3xT\,\partial/\partial T$ generates the invariants τ and $(1 + x^2)T^{2/3}$ and the invariant solution has the form $(1 + x^2)T^{2/3} = v(\tau)$. The substitution $v(\tau)$ in (3) gives

$$v' = 2/v \text{ and } v^2 = 4\tau + C,$$

or

$$(1+x^2)^2T^{4/3} = 4\tau + C. \quad (26)$$

Finally we note that in view of the complexity of the calculations of $v(\mu)$ in the structure of the invariant solutions corresponding to some of the operators listed in (4) through (7), we do not give all of the results here. However, it was shown in examples 1-9 that the kinematic description of the process of unsteady heat conduction in terms of (3) can be used to determine the function $v(\mu)$, which can be a very difficult problem using the "traditional" approach based on equation (1).

Secondly, we note that the invariant solutions obtained here, which are related to the intermediate asymptotic solutions of [6-8], contain important information on the behavior of the general solutions of boundary-value problems for the nonlinear heat-conduction equation, both for the case of fixed boundaries (or moving boundaries with a known form of the motion) as well as for the case where the motion of the boundary in time is found from an additional condition (the Stefan condition) in the case of a phase transition of the material.

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FINITE-DIFFERENCE METHOD FOR SOLVING A ONE-DIMENSIONAL NONSTATIONARY PROBLEM OF RADIATIVE—CONDUCTIVE HEAT TRANSFER

Yu. V. Lipovtsev and O. N. Tret'yakova

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An algorithm and examples of the solution of problems of complex heat transfer are given.

In this work, an effective method is offered for solving one-dimensional nonstationary boundary-value problems of radiative—conductive heat transfer with the exact equations for radiative transfer [1]. In this paper, results are presented on further development of works [2, 3], and examples of calculations and comparisons with known results of other authors are given.

We consider a flat layer of an emitting, absorbing, and anisotropically scattering medium with optically smooth or diffusely reflecting partially transparent surfaces. Initially, a nonuniformly heated layer is placed in the medium, the temperature and the coefficient of heat emission of which change according to a given law. Under the condition of azimuthal symmetry, we write the equations of radiative transfer in the form [1]

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